

SEGAL OBJECTS AND THE GROTHENDIECK CONSTRUCTION

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ABSTRACT. We discuss right fibrations in the ∞ -categorical context of Segal objects in a category \mathcal{V} and prove some basic results about these.

Right (left) and (co)Cartesian fibrations form the backbone of ∞ -category theory [Joy08], [Lur09]. The purpose of this note is to investigate them from the point of view of Rezk's Segal spaces and, more generally, Segal objects in an enriching category \mathcal{V} .

Let B be a Segal space, not necessarily complete (for definitions, see section 1). We refer to B_0 as the space of objects, B_1 as the space of morphisms, d_0 is the operator *source* and d_1 is the operator *target*. A **right fibration** over B is a Segal space X together with a map $X \rightarrow B$ having the property that the square

$$(0.1) \quad \begin{array}{ccc} X_0 & \xleftarrow{d_1} & X_1 \\ \downarrow & & \downarrow \\ B_0 & \xleftarrow{d_1} & B_1 \end{array}$$

is homotopy cartesian. A map between right fibrations over B is a weak equivalence if it is a weak equivalence on 0-simplices (alias object spaces).

After defining a model structure whose homotopy theory is that of right fibrations, we prove the following strictification result.

Theorem A. *Let \mathcal{C} be a category (possibly simplicially enriched or, more generally, internal to spaces). The Grothendieck construction (see section 1) is a simplicial Quillen equivalence between the right fibration model structure on the category $s\mathcal{S}_{/NC}$ of simplicial spaces over the nerve of \mathcal{C} and the projective model structure on $\text{Fun}(\mathcal{C}^{op}, \mathcal{S})$.*

The next step in generality are Cartesian fibrations. Still for a fixed Segal space B (which we view as a bisimplicial space with (m, n) -simplices B_m), we say that a map $X \rightarrow B$ of bisimplicial spaces is a **Cartesian fibration** if the simplicial spaces $X_{\bullet, n}$ and $X_{m, \bullet}$ are Segal spaces for all n and m , and the map

$$X_{\bullet, n} \rightarrow B_{\bullet, n}$$

is a right fibration for every n . Moreover, a map $X \rightarrow Y$ between Cartesian fibrations is a weak equivalence if it is a Dwyer-Kan equivalence on 0-simplices $X_{0, \bullet} \rightarrow Y_{0, \bullet}$.

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After defining a model structure whose underlying homotopy theory is that of Cartesian fibrations, we prove the analogous strictification statement below.

Theorem B. *The Grothendieck construction is a right Quillen equivalence between the right fibration model structure on $ss\mathcal{S}_{/NC}$ and the projective model structure on $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$. It is moreover an equivalence of $(\infty, 2)$ -categories.*

Here Cat_∞ refers to a category of $(\infty, 1)$ -categories, which for concreteness we take to be Rezk's model category of complete Segal spaces.

Remark 0.1. Theorems A and B have counterparts in the quasi-categorical setting that have appeared in the work of Joyal [Joy08], Lurie [Lur09] and Heuts-Moerdijk [HM13], [HM16], among others. An exception is the internal version of these theorems, as far as we know.

Remark 0.2. The definition of a right fibration of (complete) Segal spaces is also not new nor unexpected; Michael Shulman has mentioned a similar definition in unpublished notes of Charles Rezk, and Illan Barnea pointed out to us that another definition has appeared (independently) in [KV14].

The definition of a Cartesian fibration above has a different flavor to the construction that usual bears that name (in categories or quasi-categories). Theorem B argues that from a certain point of view both try to achieve the same (more on this in remark 3.3).

0.1. Outline. In section 1, after some preliminary work, we prove Theorem A. We also show that right fibrations are fibrations in the *complete* Segal space model structure, and directly relate our definition of a right fibration to the quasi-categorical one. In section 2 we define the homotopy theory underlying right fibration objects in a more general category \mathcal{V} , and in section 3 we apply that to define the homotopy theory of Cartesian fibrations and prove Theorem B. In section 4 we describe (co)limits of right fibrations. In section 5 we give an explicit formula for Kan extensions of a right fibration in spaces and then show that the homotopy theory of right fibrations is invariant under (Dwyer-Kan) equivalences of the base.

Notation. As customary, we denote the simplex category by Δ . Objects in Δ are finite totally ordered sets $[n] = \{0 < 1 < \dots < n\}$, for $n \geq 0$, and morphisms are order-preserving maps. Often we implicitly view $[n]$ as a category $0 \rightarrow 1 \rightarrow \dots \rightarrow n$.

Throughout, \mathcal{S} denotes the category of spaces. (*space* may be used as a synonym for *simplicial set*.) We write $s\mathcal{S}$ for the category of simplicial spaces, or more generally, $s\mathcal{V}$ for the category of simplicial objects in a category \mathcal{V} , that is, contravariant functors from Δ to \mathcal{V} . We frequently and unapologetically regard a simplicial set X as a *discrete* simplicial space with n -simplices given by the constant simplicial set X_n . In particular, we also use the notation $\Delta[n]$ for the (representable) discrete simplicial space (which Rezk refers to as $F[n]$). For a fixed object C in $s\mathcal{V}$, we write $s\mathcal{V}_{/C}$ for the category of simplicial objects over C .

For a category \mathcal{C} , which may be enriched or internal in spaces, we denote by NC the usual nerve of \mathcal{C} , i.e. the simplicial *space* whose space of 0-simplices is the space $\mathrm{ob}(\mathcal{C})$ of objects of \mathcal{C} , the space of 1-simplices is the space $\mathrm{mor}(\mathcal{C})$ of morphisms of \mathcal{C} and whose space of n -simplices is the space of n composable morphisms of \mathcal{C} , i.e. the limit (iterated pullback) of the diagram

$$\mathrm{mor}(\mathcal{C}) \xrightarrow{d_1} \mathrm{ob}(\mathcal{C}) \xleftarrow{d_0} \mathrm{mor}(\mathcal{C}) \xrightarrow{d_1} \dots \xleftarrow{d_0} \mathrm{mor}(\mathcal{C})$$

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1. LEFT AND RIGHT FIBRATIONS

1.1. Segal spaces. The following definition is due to Charles Rezk. It is a model for the umbrella term ∞ -category.

Definition 1.1 ([Rez01]). A **Segal space** is a simplicial space X such that, for each $n \geq 2$, the map

$$(1.1) \quad X_n \xrightarrow{(\alpha_0^*, \dots, \alpha_{n-1}^*)} \operatorname{holim}(X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \dots \xleftarrow{d_0} X_1)$$

is a weak equivalence of spaces, where α_i is the map $\Delta[1] \rightarrow \Delta[n]$ sending $0 < 1$ to $i < i + 1$.

Elements of X_0 are called *objects* and elements of X_1 are called *morphisms*.

Example 1.2. The nerve of a simplicially enriched category is a Segal space (the source-target maps are surjective maps onto a discrete space, hence fibrations). The nerve of an internal category in spaces is a Segal space if either the source or target map is a fibration.

Definition 1.3. Let X be a Segal space. Given two objects $x, y \in X_0$, the space of morphisms $\operatorname{mor}_X(x, y)$ is defined as the homotopy fiber, over (x, y) , of

$$X_1 \xrightarrow{(d_0, d_1)} X_0 \times X_0$$

Two maps $f, g \in \operatorname{mor}_X(x, y)$ are said to be homotopic if they lie in the same component. The *homotopy category* of X is the category $\operatorname{Ho}(X)$ with objects X_0 and morphisms $\operatorname{map}_{\operatorname{Ho}(X)}(x, y) := \pi_0 \operatorname{mor}_X(x, y)$. A map $f \in \operatorname{mor}_X(x, y)$ is a *homotopy equivalence* if it admits a left and right inverse in $\operatorname{Ho}(X)$.

The image of the space of homotopy equivalences in X_1 is denoted X_1^{he} . It is necessarily a union of components since if a map can be connected by a path to a homotopy equivalence, then it must itself be a homotopy equivalence [Rez01, Lemma 5.8].

Definition 1.4. A simplicial map $i : C \rightarrow D$ between Segal spaces is a *Dwyer-Kan equivalence* if it is fully faithful, i.e. for each pair of objects $x, y \in C_0$, the induced map

$$\operatorname{mor}_C(x, y) \rightarrow \operatorname{mor}_D(f(x), f(y))$$

is a weak equivalence of spaces, and $\operatorname{Ho}(f) : \operatorname{Ho}(X) \rightarrow \operatorname{Ho}(Y)$ is essentially surjective.

Example 1.5. Suppose $(\mathcal{C}, \mathcal{W})$ is a category \mathcal{C} with a subcategory \mathcal{W} containing all objects. Such a pair $(\mathcal{C}, \mathcal{W})$ goes by the name of *relative category* in [BK12b]. From a relative category $(\mathcal{C}, \mathcal{W})$, Rezk defines a Segal space $N(\mathcal{C}, \mathcal{W})$, the *Rezk nerve* (Rezk calls it the *classification diagram*) which retains the homotopical information of the pair. The simplicial set $N(\mathcal{C}, \mathcal{W})_m$ of m -simplices is defined as the nerve of the category of functors $[n] \times [m] \rightarrow \mathcal{C}$ which send all the arrows in the n -direction to \mathcal{W} .

An older, closely related construction is the *hammock localization* of Dwyer-Kan [DK80]. This is a simplicially enriched category $L^H \mathcal{C}$ which is a simplicial

enhancement of the classical localization, in the sense that the category obtained by taking π_0 of the morphism spaces in $L^H\mathcal{C}$ is the classical localization of \mathcal{C} with respect to \mathcal{W} . If \mathcal{C} happens to be a simplicially enriched model category, then the morphism spaces in $L^H\mathcal{C}$ are identified with the derived mapping spaces computed relative to the model structure on \mathcal{C} .

In [BK12a], Barwick and Kan show that the Rezk nerve and the hammock localization produce equivalent Segal spaces: $N(L^H\mathcal{C})$, the (usual) nerve of $L^H\mathcal{C}$, is naturally Dwyer-Kan equivalent to $N(\mathcal{C}, \mathcal{W})$ ¹.

1.2. Right fibrations. Let F be a *contravariant* functor from \mathcal{C} to spaces (possibly enriched). The *Grothendieck construction* of F is a simplicial space $\mathcal{G}(F)$ over NC defined as

$$\mathcal{G}(F)_n := \coprod_{c_0, \dots, c_n} \text{map}_{\mathcal{C}}(c_0, c_1) \times \cdots \times \text{map}_{\mathcal{C}}(c_{n-1}, c_n) \times F(c_n)$$

Degeneracies select the identity maps. Face maps encode functoriality. For example, the maps d_0 and d_1

$$\coprod_{c_0, c_1} \text{map}_{\mathcal{C}}(c_0, c_1) \times F(c_1) \rightarrow \coprod_{c_0} F(c_0)$$

are composition and projection, respectively (remember F is contravariant). As defined, $\mathcal{G}(F)$ is the nerve of an internal category in spaces (i.e. category object in spaces), and so is $\mathcal{G}(F)$ a Segal space (the target map is a projection, hence a fibration). There is reference functor from $\mathcal{G}(F)$ to the nerve of \mathcal{C} and it is easy to see that \mathcal{G} is fully faithful as a functor $\text{Fun}(\mathcal{C}^{op}, \mathcal{S}) \rightarrow s\mathcal{S}_{/NC}$.

Clearly, the Grothendieck construction $\mathcal{G}(F) \rightarrow NC$ of a *contravariant* functor F is a right fibration.

Remark 1.6. The dual notion to a right fibration is a *left fibration*. A map $p : X \rightarrow B$ in between Segal spaces is a *left fibration* if $X^{op} \rightarrow B^{op}$ is a right fibration². Alternatively, p is a left fibration if the variant of the square (1.3) with d_1 replaced by d_0 , is homotopy cartesian. The Grothendieck construction of a *covariant* functor is a *left* fibration.

Proposition 1.7. *Suppose that the square (1.3) is homotopy cartesian for a map of simplicial spaces $X \rightarrow B$ and B is a Segal space. The following statements are equivalent.*

- (1) X is a Segal space (i.e. $X \rightarrow B$ is a right fibration)

¹More precisely, they show that the Rezk nerve of $(\mathcal{C}, \mathcal{W})$ is Dwyer-Kan equivalent to the Rezk nerve of the relativization of $L^H\mathcal{C}$ [BK12a, Proposition 3.1] and in [BK12a, Proposition 1.1] that the Rezk nerve of the relativization of a simplicially enriched category A is naturally weakly equivalent to $N(A)$, the (usual) nerve of A . (Warning: Barwick-Kan use the letter N for the Rezk nerve, and the letter Z for what we refer to as the nerve.)

²By definition, the opposite of a Segal space Y is the Segal space Y^{op} obtained by reversing the order of the simplicial maps, i.e. $Y_n^{op} := Y_n$, $d_i : Y_n^{op} \rightarrow Y_{n-1}^{op}$ is defined as $d_{n-i} : Y_n \rightarrow Y_{n-1}$ and similarly for s_i .

(2) For each $n \geq 1$, the diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{a_n^*} & X_n \\ \downarrow & & \downarrow \\ B_0 & \xleftarrow{a_n^*} & B_n \end{array}$$

is homotopy cartesian, where $a_n : [0] \rightarrow [n]$ is the map sending 0 to n .

(3) For each $n \geq 1$, the diagram

$$\begin{array}{ccc} X_{n-1} & \xleftarrow{d_0} & X_n \\ \downarrow & & \downarrow \\ B_{n-1} & \xleftarrow{d_0} & B_n \end{array}$$

is homotopy cartesian, where d_0 refers to the face map induced by the map $d^0 : [n-1] \rightarrow [n]$ which misses 0.

Proof. This is rather formal and it works for any model category which, like \mathcal{S} , is right proper. Suppose (1) holds. We show (2) holds by induction on n . The case $n = 1$ holds by assumption. Suppose, for the induction hypothesis, that (2) holds for $n - 1$. Factor the square in (2) as in the diagram below.

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{n-1} \times_{X_0}^h X_1 & \longleftarrow & X_n \\ \downarrow & & \downarrow & & \downarrow \\ B_0 & \longleftarrow & B_{n-1} \times_{B_0}^h B_1 & \longleftarrow & B_n \end{array}$$

(rows induced by $\Delta[0] \rightarrow \Delta[n-1] \amalg_{\Delta[0]} \Delta[1] \rightarrow \Delta[n]$; the first map is given by the inclusion $\alpha_1 = d^1 : \Delta[0] \rightarrow \Delta[1]$, the second is the Segal map)

The right-hand square is homotopy cartesian since the horizontal maps are weak equivalences (using the assumption that X and B are Segal spaces). The left-hand square is also homotopy cartesian; this follows by commuting homotopy pullbacks, the induction hypothesis and the assumption that $X \rightarrow B$ is a right fibration.

The equivalence between (2) and (3) is straightforward.

It remains to show that (3) implies (1). We will verify the Segal condition by showing that the commutative square

$$(1.2) \quad \begin{array}{ccc} X_n & \xrightarrow{d_n} & X_{n-1} \\ d_0 \downarrow & & \downarrow d_0 \\ X_{n-1} & \xrightarrow{d_{n-1}} & X_{n-2} \end{array}$$

is homotopy cartesian, for each $n \geq 2$. It is not hard to deduce that this is an equivalent formulation of Segal's condition. Now prolong the above square to the

diagram

$$\begin{array}{ccccc}
 X_n & \xrightarrow{d_n} & X_{n-1} & \longrightarrow & B_{n-1} \\
 d_0 \downarrow & & \downarrow d_0 & & \downarrow d_0 \\
 X_{n-1} & \xrightarrow{d_{n-1}} & X_{n-2} & \longrightarrow & B_{n-2}
 \end{array}$$

Here, the right-hand square is homotopy cartesian by assumption. So it remains to show that the outer rectangle is homotopy cartesian. That outer rectangle is identified with the outer rectangle in the diagram:

$$\begin{array}{ccccc}
 X_n & \longrightarrow & B_n & \xrightarrow{d_n} & B_{n-1} \\
 d_0 \downarrow & & \downarrow d_0 & & \downarrow d_0 \\
 X_{n-1} & \longrightarrow & B_{n-1} & \xrightarrow{d_{n-1}} & B_{n-2}
 \end{array}$$

But both squares in this diagram are homotopy cartesian, the left-hand square by assumption (2) and the right-hand square since B is a Segal space. \square

1.3. Homotopy theory of right fibrations. We begin by recalling the Segal space model structure on the category of simplicial spaces. There is, throughout the paper, an underlying model structure on the category of simplicial spaces with *levelwise* weak equivalences from which all the other model structures are obtained by localization. Being a functor category, there are two standard choices for this model structure depending on whether we ask the cofibrations or the fibrations to be given levelwise (usually called injective and projective model structure, respectively). But there is also a third model structure (the Reedy model structure) which sits in between the other two (and is the one Rezk uses). It is well known that the identity map is a Quillen equivalence between the three model structures. Because of this, whenever making a distinction is not needed, we refer to either generically as the model structure on *simplicial spaces*. In other words, whenever we say that a map is a weak equivalence/fibration/cofibration of *simplicial spaces* we mean that in either one of the three senses above (but the meaning should of course be fixed, at least locally).

Proposition 1.8. (*Rezk*) *There is a left proper, simplicial model structure structure on $s\mathcal{S}$ – called the **Segal space model structure** – uniquely characterised by the following properties:*

- *An object X is fibrant if it is a Segal space (and is fibrant as a simplicial space).*
- *A morphism $X \rightarrow Y$ between (fibrant) Segal spaces is a weak equivalence if $X_n \rightarrow Y_n$ is a weak equivalence for each $n \geq 0$.*
- *A morphism is a cofibration if it is a cofibration of simplicial spaces.*

More generally, a morphism $f : X \rightarrow Y$ between any two simplicial spaces is a weak equivalence if the induced map on derived mapping spaces

$$\mathbb{R}\mathrm{map}(f, Z) : \mathbb{R}\mathrm{map}(Y, Z) \rightarrow \mathbb{R}\mathrm{map}(X, Z)$$

(computed with respect to the levelwise model structure) is a weak equivalence of spaces for every Segal space Z .

Remark 1.9. (Alternative characterization of the Segal condition) For each $n \geq 2$, the maps

$$(\alpha_0, \dots, \alpha_{n-1}) : \Delta[1] \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} \Delta[1] \hookrightarrow \Delta[n]$$

is a weak equivalence by definition of the Segal space model structure. Here is a folklore statement that resonates with the definition of a quasi-category. For $n \geq 2$, let $\iota : \Lambda^k[n] \hookrightarrow \Delta[n]$ be an inner horn inclusion, i.e. $0 < k < n$. Then ι is a weak equivalence in the Segal space model structure. One can show this inductively on n . The case $n = 2$ is already part of the Segal condition. For higher n , factor the map $(\alpha_0, \dots, \alpha_{n-1})$ as

$$\Delta[1] \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} \Delta[1] \hookrightarrow \Lambda^k[n] \rightarrow \Delta[n]$$

The left-hand map can be given as a composition of inner horn extensions, i.e. pushouts along inner horn inclusions. These inner horn inclusions are all of *lower* dimension and so are weak equivalences by induction. It follows that each extension is a pushout along a weak equivalence (which is also a cofibration in the injective structure) and so a weak equivalence. By two-out-of-three, we conclude that the right-hand map in the display is a weak equivalence.

Proposition 1.10. *Fix a Segal space B . There is a left proper, simplicial model structure on $s\mathcal{S}_B$ – called the **right fibration model structure** – uniquely characterised by the following properties:*

- *An object $X \rightarrow B$ is fibrant if it is a right fibration and a fibration of simplicial spaces.*
- *A morphism $X \rightarrow B$ between right fibrations over Z is a weak equivalence if $X_0 \rightarrow Y_0$ is a weak equivalence, i.e. if the maps of homotopy fibers $X_b \rightarrow Y_b$ are weak equivalences for each $b \in B_0$.*
- *A morphism is a cofibration if it is a cofibration of simplicial spaces.*

More generally, a morphism $f : X \rightarrow Y$ between any two simplicial spaces over B is a weak equivalence if the induced map on derived mapping spaces

$$\mathbb{R}\mathrm{map}_B(f, W) : \mathbb{R}\mathrm{map}_B(Y, W) \rightarrow \mathbb{R}\mathrm{map}_B(X, W)$$

(computed with respect to the levelwise model structure) is a weak equivalence of spaces for every fibrant object $W \rightarrow B$.

Proof. Start with the Segal space model structure on simplicial spaces. Take the overcategory model structure (a map over B is a co / fibration / weak equivalence if it is a co / fibration / weak equivalence of Segal spaces). Note that a fibrant object is a fibration in the Segal space model structure (which, since B is fibrant there, is equivalent to X being Segal and the map being a levelwise fibration). Then take the left Bousfield localization with respect to the set of maps

$$\{\mathbb{L}\Delta[0] \xrightarrow{\mathbb{L}d_1} \mathbb{L}\Delta[1] \xrightarrow{f} B : f \in B_1\}$$

where $\mathbb{L}d_1$ refers to a replacement of d_1 by a cofibration in the underlying model structure of simplicial spaces (note that no replacement is needed for the injective or Reedy model structures.) \square

Proposition 1.11. *Right fibrations are stable under base change.*

Proof. Suppose $X \rightarrow Z$ is a right fibration and $Z' \rightarrow Z$ a map. Let X' denote the homotopy pullback $X \times_Z^h Z'$. We have a commutative diagram

$$(1.3) \quad \begin{array}{ccccc} X'_1 & \xrightarrow{d_1} & X'_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ X'_1 & \xrightarrow{d_1} & Z'_0 & \longrightarrow & Z_0 \end{array}$$

The right-hand square is homotopy cartesian by assumption. The outer rectangle is also homotopy cartesian, since it is also given as the composition of two different homotopy cartesian squares. So the left-hand square is homotopy cartesian. \square

Proposition 1.12. *Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are maps between Segal spaces.*

- (1) *f and g are right fibrations $\Rightarrow gf$ is a right fibration*
- (2) *g and gf are right fibrations $\Rightarrow f$ is a right fibration*
- (3) *f and gf are right fibrations and f induces a surjection $\pi_0 X_0 \rightarrow \pi_0 Y_0 \Rightarrow g$ is a right fibration.*

Proof. (1) and (2) are straightforward. (3) follows from [dBW15, Lemma B.9]. \square

Remark 1.13. Proposition 1.7 can be rephrased as follows. For every string of morphisms $b = (b_n \leftarrow \cdots \leftarrow b_0) \in B_n$, the maps

$$\Delta[0] \xrightarrow{\alpha_n} \Delta[n] \xrightarrow{b} B \quad \text{and} \quad \Delta[n-1] \xrightarrow{d_0} \Delta[n] \xrightarrow{b} B$$

are weak equivalences in the right fibration model structure.

1.4. Right fibrations are fibrations in Rezk's model structure of complete Segal spaces.

Definition 1.14. A simplicial space X is **complete** if either (and thus all) of the maps d_0, d_1, s_0 induce a weak equivalence $X_1^{\text{he}} \simeq X_0$.

Remark 1.15. The following reformulation of the completeness condition is useful. Let E denote the (nerve of the) groupoid with two objects x, y and exactly two (non-identity) isomorphisms $x \rightarrow y$ and $y \rightarrow x$. A simplicial space X is complete if and only if the map

$$\mathbb{R}\text{map}(E, X) \rightarrow \mathbb{R}\text{map}(\Delta[0], X)$$

induced by any of the (two) maps $\Delta[0] \rightarrow E$, is a weak equivalence. The *complete* Segal space model structure [Rez01] is constructed by localizing the Segal space model structure with respect to these maps. The weak equivalences between Segal spaces (not necessarily complete) become the Dwyer-Kan equivalences.

Definition 1.16. A simplicial space X over B is **fiberwise complete** if the square

$$\begin{array}{ccc} X_0 & \xleftarrow{d_1} & X_1^{\text{he}} \\ \downarrow & & \downarrow \\ B_0 & \xleftarrow{d_1} & B_1^{\text{he}} \end{array}$$

is homotopy cartesian.

Remark 1.17. A levelwise fibration $p : X \rightarrow B$ is a fiberwise complete Segal space if and only if every diagram

$$(1.4) \quad \begin{array}{ccc} \Delta[0] & \longrightarrow & X \\ \text{target} \downarrow & & \downarrow \\ E & \longrightarrow & B \end{array}$$

has a contractible space of lifts for each pair of horizontal maps. More precisely, for each $x \in B_1^{\text{he}}$, the map

$$(1.5) \quad \mathbb{R}\text{map}_B(f, X) : \mathbb{R}\text{map}_B(E, X) \rightarrow \mathbb{R}\text{map}_B(\Delta[0], X)$$

induced by $\Delta[0] \rightarrow E \xrightarrow{x} B$ is a weak equivalence of spaces.

Proposition 1.18. *Let B be a Segal space. A map $p : X \rightarrow B$ is a fibration in the complete Segal space model structure if and only if p is fiberwise complete and a levelwise fibration. In particular, a right fibration is a fibration in the complete Segal space model structure.*

Proof. The first assertion is shown in [dBW15, Appendix B]. To prove the second assertion, we suppose $X \rightarrow B$ is a right fibration and will show that X is fiberwise complete over B .

Let Φ_x denote the homotopy fiber of $d_1 : X_1^{\text{he}} \rightarrow X_0$ over an object x , and Φ_b denote the homotopy fiber of $d_1 : B_1^{\text{he}} \rightarrow B_0$ over an object $b = p(x)$. The space X_1^{he} may be described as $X_3 \times_{X_1 \times X_1}^h X_0 \times X_0$, where the maps $[1] \rightarrow [3]$ are: $(0 < 1)$ is sent to $(0 < 2)$ and $(1 < 3)$ resp. Thus, the map of fibers is identified with the map

$$X_3 \times_{X_1 \times X_1}^h X_0 \times \{x\} \rightarrow B_3 \times_{B_1 \times B_1}^h B_0 \times \{b\}$$

The result now follows by inspection of fibers, using the right fibration assumption. \square

Corollary 1.19. *If $X \rightarrow B$ is a right fibration and B is a complete Segal space, then X is a complete Segal space.*

Remark 1.20. Proposition 1.18 implies that the left Bousfield localization of the (overcategory) complete Segal space model structure with respect to the same maps as in Proposition 1.10 defines the same model category of right fibrations.

Proposition 1.21. *Suppose $f : X \rightarrow Y$ is a Dwyer-Kan equivalence between Segal spaces over B . Then f is a weak equivalence in the right fibration model structure. If in addition f is a right fibration then f is a levelwise weak equivalence.*

Proof. Dwyer-Kan equivalences are the weak equivalences between Segal spaces in the complete Segal space model structure, and the right fibration model structure on B is a left Bousfield localization of the overcategory of the complete Segal space model structure, so the first claim follows.

As for the second claim, using Proposition 1.18 the assumption is that f is an acyclic fibration in the complete Segal space model structure. But in any left Bousfield localization, the set of acyclic fibrations coincides with the set of acyclic fibrations of the underlying model structure (which in this case is the underlying model structure on simplicial spaces). \square

1.5. Comparison with quasi-categories. Recall that a space (simplicial set) X is viewed as a discrete simplicial space (bisimplicial set) by taking the pullback along $p_1 : \Delta \times \Delta \rightarrow \Delta$, the projection onto the first factor. (The space (set) of n -simplices of p_1^*X is X_n , viewed as a constant simplicial set.) The functor p_1 has a left adjoint, i_1 , given by $i_1([n]) = ([n], [0])$, so there is an adjunction

$$p_1^* : \mathcal{S} \rightleftarrows s\mathcal{S} : i_1^*X$$

The set of n -simplices of i_1^*X is the set of 0-simplices of X_n , the space of n -simplices of X . By [JT07], the pair (p_1^*, i_1^*) forms a Quillen equivalence between the model structure for quasi-categories (alias the *Joyal* model structure) and the model structure for complete Segal spaces (with underlying Reedy model structure on simplicial spaces). We can use this to show that the notion of right fibration for Segal spaces agrees with the quasi-categorical one:

Theorem 1.22. *Let B be a (Reedy fibrant) complete Segal space. Then (i_1^*, p_1^*) forms a Quillen equivalence between $s\mathcal{S}_B$ and $\mathcal{S}_{/i_1^*B}$, both equipped with the right fibration model structure.*

Proof. According to [HM13], when Y is a quasi-category, the right fibration model structure on \mathcal{S}_Y is obtained by left Bousfield localization of the overcategory of the Joyal model structure, with respect to right horn inclusions, i.e. maps $\Lambda^n[n] \hookrightarrow \Delta[n] \rightarrow Y$ for each $n \geq 1$, where $\Lambda^n[n]$ is the union of all the faces except d^n . For a complete Segal space B , we have established (c.f. remark 1.20) that the right fibration model structure on $s\mathcal{S}_B$ can be obtained as the left Bousfield localization of the overcategory of the complete Segal space model structure with respect to the maps $p_1^*\Delta[0] \rightarrow p_1^*\Delta[1] \rightarrow Y$ (left-hand map is d_0).

The claim is that the overcategory model structures are being localized at the same set of maps. In other words, the maps $p_1^*(\Lambda^n[n]) \hookrightarrow p_1^*(\Delta[n])$ (over $\mathbb{L}p_1^*i_1^*B \simeq B$) are weak equivalences in the complete Segal space side. For $n = 1$ there is nothing to prove. For higher n , factor d_0 as

$$\Delta[n-1] \rightarrow \Lambda^n[n] \rightarrow \Delta[n]$$

(Here, and in line with the rest of the paper, we have omitted p_1^* from the notation.) The composite map is a weak equivalence by Proposition 1.18. The left-hand map can be given as a composition of lower dimensional right horn extensions (that is, pushouts along right horn inclusions $\Lambda^k[k] \rightarrow \Delta[k]$ for $k < n$), each of which is an acyclic cofibration by induction. Therefore, the left-hand map in the display is a weak equivalence and so, by two-out-of-three, the right-hand map is a weak equivalence. \square

Remark 1.23. The condition that B is *complete* can be dropped. Indeed, a Segal space B which is not necessarily complete can be completed: in [Rez01], Rezk constructs a Dwyer-Kan equivalence $B \rightarrow B^\sharp$ to a *complete* Segal space B^\sharp . The homotopy theories of right fibrations over B and B^\sharp are equivalent (this will be proved in corollary 5.7). Therefore, by the theorem above, the homotopy theory of right fibrations over B is equivalent to the one of quasi-categorical right fibrations over $i_1^*B^\sharp$.

1.6. Terminal objects and the generalized Yoneda lemma.

Definition 1.24. Let X and Y be simplicial spaces. The (derived) internal hom construction X^Y is the simplicial space with n -simplices given by

$$\mathbb{R}\mathrm{map}(\Delta[n] \times Y, X)$$

where $\mathbb{R}\mathrm{map}$ denotes the derived mapping space in the levelwise model structure (henceforth, this will be the standard interpretation, unless mentioned otherwise). To make sure this definition is functorial in $[n]$, one should choose *functorial* cofibrant/fibrant replacement functors in $s\mathcal{S}$.

Remark 1.25. The construction above *may not be* the derived functor (in a model categorical sense) of the usual internal hom functor in simplicial spaces, e.g. if the projective model structure is taken. If one works with the injective model structure or the Reedy one (as Rezk does), then it is. That is a consequence of the fact that these two model structures are *closed* (see [Rez01]). We'll be cavalier about this point and simply use the definition above. The reason for that is we will often want to use the projective model structure.

Definition 1.26 (Generalized overcategory). Let X be a Segal space. For a space K and map $\alpha : K \rightarrow X_0$, define X/K to be the simplicial space obtained as the homotopy pullback of

$$X^{\Delta[1]} \xrightarrow{d_1} X \xleftarrow{\alpha} \Delta^0 \times K$$

Remark 1.27. When $K = X_0$ and α is the identity map, X/X_0 is the Segal space whose objects are morphisms $x \rightarrow y$ in X . A morphism is a string $x \rightarrow x' \rightarrow y$ (its source is the composite $x \rightarrow y$ and its target is $x' \rightarrow y$). Note that this construction is not what is often referred to as the arrow category. Rather, it is a overcategory-type construction, as exemplified by the case $K = *$ and α the map selecting an object x .

Proposition 1.28. *The map $d_0 : X/K \rightarrow X$, induced by $d_0 : X^{\Delta[1]} \rightarrow X$, is a right fibration.*

Proof. Rezk shows [Rez01] that $X^{\Delta[1]}$ is a Segal space. This implies that X/K is a Segal space, so we only need to verify the fibration condition (1.3). We will do this for $Y := X/X_0$. The general case then follows, using that $X/K \simeq Y \times_{X_0} K$.

By construction, Y_0 is identified with X_1 . Let us describe Y_1 in words, first. It is the space of commutative squares in X , i.e. $gf = g'f'$, with the extra condition that g' is an identity morphism. This last condition obviously forces the equality $f' = gf$. That is, Y_1 is identified with the space of two composable morphisms in X , alias X_2 . More formally, use the decomposition

$$\Delta[1] \times \Delta[1] = \Delta[2] \amalg_{\Delta[1]} \Delta[2]$$

of the square in two triangles, to write Y_1 as the homotopy limit of

$$X_2 \xrightarrow{d_1} X_1 \xleftarrow{d_1} X_2 \xrightarrow{d_2} X_1 \xleftarrow{s_0} X_0$$

Since X is a Segal space, we have that $X_2 \times_{X_1} X_0 \simeq X_1$ and hence $Y_1 \simeq X_2$, as claimed. \square

Definition 1.29. An object $x \in X$ is *terminal* if $d_0 : X/x \rightarrow X$ is a weak equivalence of simplicial spaces.

This is equivalent to the condition that the homotopy fiber of the target map $d_1 : X_1 \rightarrow X_0$ over x be weakly equivalent to X_0 .

The crucial observation is

Lemma 1.30. *Let $p : X \rightarrow B$ be a simplicial space over B and suppose X has a terminal object x . Then for every right fibration $q : Y \rightarrow B$ the map*

$$(1.6) \quad x^* : \mathbb{R}\mathrm{map}_B(X, Y) \rightarrow \mathbb{R}\mathrm{map}_B(\Delta[0], Y)$$

given by precomposition with $x : \Delta[0] \rightarrow X$ is a weak equivalence. That is to say, the inclusion $\Delta[0] \xrightarrow{x} X$ (over B) is a weak equivalence in the right fibration model structure.

The following immediate corollary may be regarded as an $(\infty, 1)$ -categorical form of the Yoneda Lemma.

Lemma 1.31 (Yoneda Lemma). *The inclusion $\Delta[0] \rightarrow Z/z$ over Z which selects $id_z := s_0(z)$ is a weak equivalence in the right fibration model structure.*

Remark 1.32. We will use the following simple observation repeatedly in the proof below. Suppose B is a fibrant simplicial space. By definition, the derived mapping space $\mathbb{R}\mathrm{map}_B(X, Y)$ (in the right fibration sense) is weakly equivalent to $\mathrm{map}_B(X^c, Y^f)$ where $Y^f \rightarrow B$ is a fibrant replacement (a right fibration which is also a fibration of simplicial spaces) and X^c a cofibrant replacement (cofibrant as a simplicial space). The latter space is of course the pullback (alias fiber) of

$$* \rightarrow \mathrm{map}(X^c, B) \leftarrow \mathrm{map}(X^c, Y^f)$$

But the right hand-map is a fibration, and both mapping spaces in the display are weakly equivalent to the derived mapping spaces (with respect to the levelwise model structure). Thus, for a right fibration $Y \rightarrow B$, $\mathbb{R}\mathrm{map}_B(X, Y)$ is naturally weakly equivalent to the homotopy pullback of

$$* \rightarrow \mathbb{R}\mathrm{map}(X, B) \leftarrow \mathbb{R}\mathrm{map}(X, Y)$$

Proof of Lemma 1.30. Without loss of generality, we may assume that $Y \rightarrow B$ is fibrant (a right fibration and fibration of simplicial spaces) and X is cofibrant and B is fibrant as a simplicial space. Let $y \in Y_0$ with $q(y) = b$. Define $\mathbb{R}\mathrm{map}((X, x), (Y, y))$ as the homotopy fiber of the map

$$\mathbb{R}\mathrm{map}(X, Y) \xrightarrow{x^*} \mathbb{R}\mathrm{map}(\Delta[0], Y) = Y_0$$

over $y \in Y$, and define $\mathbb{R}\mathrm{map}((X, x), (B, b))$ similarly. By commuting homotopy limits, the homotopy fiber of (1.6) over y is identified with the homotopy fiber of

$$\mathbb{R}\mathrm{map}((X, x), (Y, y)) \rightarrow \mathbb{R}\mathrm{map}((X, x), (B, b))$$

over $p : X \rightarrow B$.

We have a commutative square

$$(1.7) \quad \begin{array}{ccc} \mathbb{R}\mathrm{map}((X, x), (Y, y)) & \longrightarrow & \mathbb{R}\mathrm{map}((X/x, id_x), (Y/y, id_y)) \\ \downarrow & & \downarrow \\ \mathbb{R}\mathrm{map}((X, x), (B, b)) & \longrightarrow & \mathbb{R}\mathrm{map}((X/x, id_x), (B/b, id_b)) \end{array}$$

(Here we follow Rezk and write $id_z \in Z_1$ for the image under s_0 of $z \in Z_0$.)

The horizontal maps are weak equivalences because x is the terminal object of X . In more detail, the composition

$\mathbb{R}\mathrm{map}((X, x), (Y, y)) \rightarrow \mathbb{R}\mathrm{map}((X/x, id_x), (Y/y, id_y)) \xrightarrow{d_0} \mathbb{R}\mathrm{map}((X/x, id_x), (Y, y))$ agrees with precomposition with $d_0 : X/x \rightarrow X$ and so it is a weak equivalence (because x is the terminal object of X). This shows that the top horizontal map in (1.7) has a homotopy left inverse.

Now let X/id_x denote the over category $(X/x)/id_x$ whose objects are 2-morphisms in X with source id_x . The composition

$$\begin{array}{ccc} \mathbb{R}\mathrm{map}((X/x, id_x), (Y/y, id_y)) & \longrightarrow & \mathbb{R}\mathrm{map}((X/x, id_x), (Y, y)) \\ & & \downarrow d_0 \\ & & \mathbb{R}\mathrm{map}((X/id_x, id_{id_x}), (Y/y, id_y)) \end{array}$$

agrees with precomposition with $d_0 : X/id_x \rightarrow X/x$ and so is a weak equivalence (because id_x is the terminal object of X/x). This shows that the top horizontal map in (1.7) has a homotopy right inverse. The same arguments apply to show that the lower horizontal map is also a weak equivalence.

Since $Y \rightarrow B$ is a right fibration, the induced map between horizontal homotopy fibers (over $y \in Y_0$ and $b \in B_0$) in the diagram

$$\begin{array}{ccc} Y_0 & \xleftarrow{d_1} & Y_1 \\ \downarrow & & \downarrow \\ B_0 & \xleftarrow{d_1} & B_1 \end{array}$$

is a weak equivalence. That is, $Y/y \xrightarrow{\sim} B/b$. Thus the right-hand map in (1.7) is also a weak equivalence. \square

Remark 1.33. There are parametrized versions of lemmas 1.30 and 1.31. Let X be a Segal space over B . If $d_0 : X/K \rightarrow X$ is a weak equivalence (i.e. K is a terminal subspace of the space objects of X) then

$$\alpha : K \times \Delta[0] \rightarrow X$$

(over B) is a weak equivalence in the right fibration model structure on $s\mathcal{S}_B$. Therefore, the map $K \times \Delta[0] \rightarrow X/K$ is a weak equivalence in the right fibration model structure on $s\mathcal{S}_X$.

1.7. Proof of Theorem A. We begin with

Proposition 1.34. *The Grothendieck construction \mathcal{G} has a simplicial left adjoint*

$$L : s\mathcal{S}_{/NC} \rightleftarrows \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) : \mathcal{G}$$

This adjunction is Quillen when both sides are equipped with the projective model structure.

Proof. Suppose X is a simplicial space over NC . Then X can be presented as the colimit (alias coequalizer) in $s\mathcal{S}$ of

$$(1.8) \quad \coprod_{[n_0]} \Delta[n_0] \times X_{n_0} \rightleftarrows \coprod_{[n_0] \rightarrow [n_1] \in \Delta} \Delta[n_0] \times X_{n_1}$$

This is a well known consequence of the Yoneda lemma. Note that the map from this colimit to X is an isomorphism over $N\mathcal{C}$.

Left adjoints preserve colimits, and so L is uniquely determined (up to isomorphism) by its value on objects of the form $\alpha : \Delta[n] \times K \rightarrow N\mathcal{C}$ for some space K . Suppose given one such object corresponding via the Yoneda lemma to a map $\alpha : K \rightarrow N\mathcal{C}_n$ from K to the space of n -composable morphisms in \mathcal{C} . Denote by α' the map $K \rightarrow N\mathcal{C}_n \rightarrow N\mathcal{C}_0$ given by applying α and then $d_1 d_2 d_3 \dots d_n$, the operation selecting the ultimate target object.

Define

$$L(\Delta[n] \times K \xrightarrow{\alpha} N\mathcal{C}) := \coprod_{c \in \text{im}(\alpha')} K_c \times \text{map}_{\mathcal{C}}(-, c)$$

where K_c is the fiber of $\alpha' : K \rightarrow N\mathcal{C}_0$ over the object c in the image of α' . Note that when $K = *$, $L(f : \Delta[n] \rightarrow N\mathcal{C})$ is the representable functor $\text{map}_{\mathcal{C}}(-, c_n)$ where c_n is the ultimate target object of f .

It is a mechanical verification using the Yoneda lemma that

$$\text{Hom}(L(\alpha), F) \cong \text{map}_{N\mathcal{C}}(K \times \Delta[n], \mathcal{G}(F))$$

for every functor F . This settles the claim that (L, \mathcal{G}) form a simplicial adjunction. Clearly, \mathcal{G} preserves fibrations and weak equivalences, and so the pair is Quillen. \square

Remark 1.35. In fact, \mathcal{G} also has a right adjoint. Let c be an object of \mathcal{C} and \mathcal{C}/c denote the overcategory (this is an internal category, with *space* of objects consisting of maps with target c). Let $N(\mathcal{C}/c) \rightarrow N(\mathcal{C})$ be the map induced by the forgetful functor $\mathcal{C}/c \rightarrow \mathcal{C}$. The functor $R : s\mathcal{S}_{/N\mathcal{C}} \rightarrow \text{PSh}(\mathcal{C})$ which to $X \in s\mathcal{S}_{/N\mathcal{C}}$ associates the presheaf $R(X)$ defined by

$$R(X)(c) := \text{map}_{N\mathcal{C}}(N(\mathcal{C}/c), X)$$

is the (simplicial) right adjoint to \mathcal{G} .

The substance of Theorem A is encompassed in the following step.

Lemma 1.36. *For each object $X \rightarrow N\mathcal{C}$, the derived unit map $X \rightarrow \mathbb{R}GL(X)$ is a weak equivalence in the right fibration model structure on the category of simplicial spaces over $N\mathcal{C}$.*

Proof of Lemma 1.36. Since \mathcal{G} preserves weak equivalences, it is enough to show that the (non-derived) unit map $X \rightarrow \mathcal{G}L(X)$ is a weak equivalence for each cofibrant simplicial space X . That is, we are required to show that for each fibrant (i.e. right fibration) Z , the induced map

$$(1.9) \quad \mathbb{R}\text{map}_{N\mathcal{C}}(\mathcal{G}L(X), Z) \rightarrow \mathbb{R}\text{map}_{N\mathcal{C}}(X, Z)$$

is a weak equivalence of spaces.

Let Y_\bullet be the canonical resolution of X by representables. Namely, Y is a simplicial object in $s\mathcal{S}_{/N\mathcal{C}}$ with simplicial space Y_k of k -simplices given by

$$\coprod_{[n_0] \rightarrow [n_1] \rightarrow \dots \rightarrow [n_k]} X_{n_k} \times \Delta[n_0]$$

The Yoneda lemma provides a map $Y_\bullet \rightarrow X$, where X is viewed as a constant simplicial object in $s\mathcal{S}_{/N\mathcal{C}}$. The induced map $|Y_\bullet| \rightarrow X$ is well known to be a levelwise weak equivalence in $s\mathcal{S}_{/N\mathcal{C}}$ (this can be proved by exhibiting a simplicial contracting homotopy).

We can thus reduce the problem to the case of representables. By Ken Brown's lemma [Hir03, 7.7.2], the left Quillen functor L preserves weak equivalences between cofibrant objects. So $L(X) \xrightarrow{\simeq} L(|Y_\bullet|)$ since X is cofibrant by hypothesis and $|Y_\bullet|$ is the geometric realisation of a Reedy cofibrant object, hence cofibrant. Therefore,

$$\mathbb{R}\mathrm{map}_{NC}(\mathcal{G}L(X), Z) \simeq \mathbb{R}\mathrm{map}_{NC}(|\mathcal{G}L(Y_\bullet)|, Z) \simeq \mathrm{Tot} \mathrm{map}_{NC}(\mathcal{G}L(Y_\bullet), Z)$$

The first equivalence is a consequence of the fact that both L and \mathcal{G} commute with geometric realisation (and colimits in general, since both are left adjoints). In the second equivalence, we have used that $|\mathcal{G}L(Y_\bullet)|$ is cofibrant since it is the geometric realisation of a Reedy cofibrant object. Similarly, $\mathbb{R}\mathrm{map}_{NC}(X, Z) \simeq \mathrm{Tot} \mathrm{map}_{NC}(Y_\bullet, Z)$.

The upshot is that it suffices to check that (1.9) is a weak equivalence for X of the form $f : \Delta[n] \rightarrow NC$ corresponding to some $f : b_0 \rightarrow \cdots \rightarrow b_n$. Because the map $\Delta[0] \rightarrow \Delta[n]$ (selecting the last vertex n) is a weak equivalence in the right fibration model structure, it suffices to prove the claim for $X = \Delta[0] \rightarrow NC$ corresponding to some object $b \in NC$. This verification occupies the remainder of the proof.

When $X = (\Delta[0] \xrightarrow{b} NC)$, the target of (1.9) is $\mathbb{R}\mathrm{map}_{NC}(\Delta[0], Z) \simeq Z_b$. We now investigate the source. Recall that $L(f) = \mathrm{map}(-, b)$. Therefore, $\mathcal{G}L(f) = NC/b$. The inclusion

$$\Delta[0] \xrightarrow{id_b} NC/b$$

over $\{b\} \in NC_0$, induces a map

$$(1.10) \quad \mathbb{R}\mathrm{map}_{NC}(NC/b, Z) \rightarrow \mathbb{R}\mathrm{map}_{NC}(\{b\}, Z) = Z_b$$

which agrees with (1.9). Clearly $\{id_b\}$ is a terminal object of NC/b . This implies, by Lemma 1.30, that (1.10) is a weak equivalence. \square

Moreover, if X is a right fibration over NC , then we immediately obtain the strictification result that X is levelwise equivalent to an actual (*strict*) presheaf:

Corollary 1.37. *Suppose X is a right fibration over NC and X is cofibrant as a simplicial space. Then the unit map $X \rightarrow \mathcal{G}L(X)$ is levelwise weak equivalence.*

Proof. Weak equivalences between fibrant objects in $s\mathcal{S}_{/NC}$ are determined levelwise, by Proposition 1.10. \square

Proof of Theorem A. We have already established that (L, \mathcal{G}) forms a Quillen pair. The unit of the induced adjunction on homotopy categories

$$(1.11) \quad \mathrm{Ho}L : \mathrm{Ho}(s\mathcal{S}_{/NC}) \rightleftarrows \mathrm{Ho}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) : \mathrm{Ho}\mathcal{G}$$

is an isomorphism by the Lemma 1.36. Moreover, \mathcal{G} reflects weak equivalences between fibrant objects. \square

1.8. Explicit description of the left adjoint.

Proposition 1.38. *There is a weak equivalence*

$$\mathrm{hocolim}_{\Delta} N(c/\mathcal{C}) \times_{NC} X \rightarrow L(X)(c)$$

natural in $c \in \mathcal{C}$. Here c/\mathcal{C} is the internal (under)category with space of objects consisting of maps in \mathcal{C} with source c . The left-hand side is clearly functorial in \mathcal{C} since a map $c \rightarrow d$ in \mathcal{C} induces a map of nerves $N(d/\mathcal{C}) \rightarrow N(c/\mathcal{C})$.

Proof. It suffices to consider the case when $X = \Delta[n] \rightarrow NC$. For each $c \in \mathcal{C}$, the simplicial space $N(c/\mathcal{C}) \times_{NC} \Delta[n]$ is equivalent to one with k -simplices given by

$$\coprod_{0 \leq i_0 \leq i_1 \leq \dots \leq i_k \leq n} \text{map}_{\mathcal{C}}(c, b_{i_0})$$

where the degeneracy maps are given by composition and the face maps are obvious. This has a simplicial contraction onto $\text{map}_{\mathcal{C}}(c, b_n)$, using the maps $b_{i_j} \rightarrow b_n$. \square

1.9. Internal version. There exists a variant of Theorem A for category objects in \mathcal{S} (alias internal categories in spaces).

Definition 1.39. Let \mathcal{C} be an internal category in spaces. An internal (contravariant) functor $F : \mathcal{C}^{op} \rightarrow \mathcal{S}$ (also called *discrete fibration*) consists of an internal category \mathcal{F} together with a functor $\mathcal{F} \rightarrow \mathcal{C}$ of internal categories (i.e., a simplicial map between nerves) with the property that the square

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\text{target}} & \mathcal{F}_0 \\ \downarrow & & \downarrow \\ \mathcal{C}_1 & \xrightarrow{\text{target}} & \mathcal{C}_0 \end{array}$$

is cartesian (alias a pullback). The subscript 1 stands for space of morphisms and the subscript 0 stands for space of objects.

We adopt the same notation as before and denote by $\text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ the (internal) category of internal functors $\mathcal{C}^{op} \rightarrow \mathcal{S}$. Horel has proved in [Hor] – under the relatively harmless condition that the target map $\mathcal{C}_1 \rightarrow \mathcal{C}_0$ is a fibration in \mathcal{S} – that the category $\text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ has a projective model structure in which a natural transformation $\mathcal{F} \rightarrow \mathcal{F}'$ over \mathcal{C} is a weak equivalence (resp. fibration) if the map $\mathcal{F}_0 \rightarrow \mathcal{F}'_0$ is a weak equivalence (resp. fibration) in \mathcal{S} .

The promised mild extension of Theorem A is

Theorem 1.40. *Let \mathcal{C} be an internal category in \mathcal{S} such that the target map $\mathcal{C}_1 \rightarrow \mathcal{C}_0$ is a fibration. Then*

$$L : s\mathcal{S}_{/NC} \rightleftarrows \text{Fun}(\mathcal{C}^{op}, \mathcal{S}) : \mathcal{G}$$

is a simplicial Quillen equivalence, where the left-hand side is equipped with the right fibration model structure. Here \mathcal{G} assigns to an internal functor $\mathcal{F} \rightarrow \mathcal{C}$ the map of simplicial spaces $N\mathcal{F} \rightarrow NC$ given by the nerve.

Proof. We need to modify slightly the construction of the left adjoint L from the proof of proposition 1.34. For an object $\alpha : \Delta[n] \times K \rightarrow NC$, declare $L(\alpha)$ to be the category with object space $L(\alpha)_0$ and morphism space $L(\alpha)_1$ defined as follows:

$$L(\alpha)_0 := \text{pullback}(\mathcal{C}_1 \xrightarrow{\text{target}} \mathcal{C}_0 \leftarrow K)$$

where the right-hand map is the composition $K \rightarrow \mathcal{C}_n \rightarrow \mathcal{C}_0$ of α with the ultimate target operator. The morphism space $L(\alpha)_1$ is defined as $\mathcal{C}_1 \times_{\mathcal{C}_0} L(\alpha)_0$. The source map

$$L(\alpha)_1 \rightarrow L(\alpha)_0$$

is induced by the composition map $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$ in \mathcal{C} . The target map is the projection. The forgetful map $L(\alpha) \rightarrow NC$ gives an internal functor $\mathcal{C}^{op} \rightarrow \mathcal{S}$. We

leave to the reader the verification that L as prescribed above (and extended to all simplicial spaces by colimits) is the left adjoint to \mathcal{G} .

The condition that the target map $\mathcal{C}_1 \rightarrow \mathcal{C}_0$ is a fibration guarantees that \mathcal{G} preserves fibrant objects and, more generally, fibrations. It is equally clear that \mathcal{G} preserves weak equivalences. So the pair (L, \mathcal{G}) is Quillen.

The argument for the enriched case goes through mostly without substantial changes to show that (L, \mathcal{G}) is a Quillen equivalence. The key step is to show that, for an object $\alpha : \Delta[0] \times K \rightarrow NC$ with K a cofibrant space, the unit map

$$\alpha \rightarrow \mathcal{G}(L(\alpha))$$

is a weak equivalence in the right fibration model structure on $s\mathcal{S}_{/NC}$. In other words, that for every right fibration $Z \rightarrow NC$, the induced map

$$\mathbb{R}\mathrm{map}_{NC}(\mathcal{G}(L(\alpha)), Z) \rightarrow \mathbb{R}\mathrm{map}_{NC}(\alpha, Z)$$

is a weak equivalence. This can be proved as before; but now both sides are weakly equivalent to $\mathbb{R}\mathrm{map}_{\mathcal{C}_0}(K, Z_0)$. For the left-hand side, this follows from lemma 1.31, or rather, remark 1.33. \square

2. SEGAL OBJECTS IN A \mathcal{V} -CATEGORY

In this section, we discuss a generalization of the previous constructions, by considering Segal objects in a category \mathcal{V} . To conform with model categorical setups, we need to impose a number of technical conditions on \mathcal{V} . We assume that \mathcal{V} is a cartesian closed model category (see e.g. [Hov99, Ch. 4], [Rez01, 2.5]) (the monoidal structure is the cartesian product). We denote the unit (i.e. the terminal object) by $*$, and we assume it is a cofibrant object. Finally, we assume that \mathcal{V} is left proper and combinatorial.

The category $s\mathcal{V}$ of simplicial objects in \mathcal{V} is tensored and cotensored over \mathcal{V} . For $X \in s\mathcal{V}$ and $K \in \mathcal{V}$, the tensor $X \times K$ is given by $[n] \mapsto X_n \times K$, and the cotensor X^K is given by $[n] \mapsto \mathrm{map}(K, X_n)$, where map here refers to the internal mapping object in \mathcal{V} . It follows that $s\mathcal{V}$ becomes a \mathcal{V} -model category (see [Hov99, 4.2.18] for a definition) for the injective and Reedy model structures, and is left proper and combinatorial (from the assumptions we made on \mathcal{V}).

The obvious definition works:

Definition 2.1. A *Segal object* in \mathcal{V} is a functor $X : \Delta^{op} \rightarrow \mathcal{V}$ such that

$$X_n \rightarrow X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1$$

is a weak equivalence in \mathcal{V} .

Definition 2.2. Let Z be a Segal object in \mathcal{V} . A map $X \rightarrow Z$ in $s\mathcal{V}$ is said to be a **right fibration** if X is a Segal object in \mathcal{V} and the square analogous to (1.3) is homotopy cartesian.

The technical assumptions on \mathcal{V} made above guarantee the existence of enriched left Bousfield localizations (see [Bar10]). So we can define a model category structure on $s\mathcal{V}_{/B}$, the category of simplicial objects in \mathcal{V} over B , whose homotopy theory is that of right fibrations over B .

Proposition 2.3. Fix a Segal object B . There is a left proper, \mathcal{V} -model structure structure on $s\mathcal{V}_{/B}$ – called the **right fibration model structure** – uniquely characterised by the following properties:

- An object $X \rightarrow B$ is fibrant if it is a right fibration and a fibration in $s\mathcal{V}$
- A morphism $X \rightarrow B$ between right fibrations over Z is a weak equivalence if $X_0 \rightarrow Y_0$ is a weak equivalence
- A morphism is a cofibration if it is a cofibration in $s\mathcal{V}$.

More generally, a morphism $f : X \rightarrow Y$ between any two simplicial objects over B is a weak equivalence if the induced map on derived mapping objects

$$\mathbb{R}\mathrm{map}_B(f, W) : \mathbb{R}\mathrm{map}_B(Y, W) \rightarrow \mathbb{R}\mathrm{map}_B(X, W)$$

(computed with respect to the levelwise model structure) is a weak equivalence in \mathcal{V} for every fibrant object $W \rightarrow B$.

We note that, by the assumptions made on \mathcal{V} , the internal mapping object $\mathrm{map}(-, -)$ is a Quillen bifunctor, and so it admits a derived functor $\mathbb{R}\mathrm{map}$.

3. CARTESIAN FIBRATIONS

In this section, we take \mathcal{V} to be Rezk's category of complete Segal spaces Cat_∞ . (In order to guarantee cartesian closedness, we need to take the underlying model structure on simplicial spaces to be the injective or Reedy model structure.) A Segal object in Cat_∞ is what one might call a 2-Segal space, that is, a bisimplicial space $X_{m,n}$ such that $X_{\bullet,n}$ is a Segal space (for each fixed $n \geq 0$) and $X_{m,\bullet}$ is a Segal space (for each fixed $m \geq 0$). This is reminiscent of a double category (a category object in categories) not of a 2-category.

A Segal space B is an example of a 2-Segal space: set $B_{m,n} = B_m$ for $m, n \geq 0$.

Remark 3.1. A 2-Segal space as defined is not a fibrant object in $s\mathcal{V}$, the missing condition being that $X_{\bullet,n}$ is complete as a Segal space.

For the rest of this section, B denotes a fixed Segal space.

Definition 3.2. A map $X \rightarrow Z$ of bisimplicial spaces is a **Cartesian fibration** if X is a 2-Segal space and the map

$$X_{\bullet,n} \rightarrow B_{\bullet,n}$$

is a right fibration (of simplicial spaces) for every n .

Remark 3.3. A word of warning about terminology: the definition above is not the literal analogue of the classical definition of a Cartesian fibration (or its quasi-categorical version), so it may be argued that it's not fully deserving of that name. For example, definition 3.2 is one categorical level higher when compared to the standard definition. We decided to keep the potentially conflicting terminology, in part due to theorem B (both perspectives give weak models for functors with values in Cat_∞) and because it is plausible that a direct comparison should exist.

3.1. Homotopy theory of Cartesian fibrations. Because of its importance, we restate Proposition 2.3 for the special case $\mathcal{V} = \mathrm{Cat}_\infty$ below.

Proposition 3.4. *There is a left proper, Cat_∞ -enriched model structure on the category of bisimplicial spaces over Z uniquely characterised by the following data.*

- An object $X \rightarrow B$ is fibrant if it is a Cartesian fibration and $X_{m,\bullet}$ is a complete Segal space for each m .
- A map $f : X \rightarrow Y$ over B is a cofibration if it is so in the underlying model structure on bisimplicial spaces.

A map $f : X \rightarrow Y$ over B is a weak equivalence if $f_{m,\bullet}$ is a weak equivalence in the complete Segal space model structure for all m , and $f_{\bullet,n}$ is a weak equivalence in the right fibration model structure for all n .

Remark 3.5. Suppose B is a Segal space (viewed as a 2-Segal space). Let X and Y be over B , but $X_{n,\bullet}$ and $Y_{n,\bullet}$ not necessarily complete. A map $f : X \rightarrow Y$ over B is a weak equivalence if and only if the map

$$f_{0,\bullet} : X_{0,\bullet} \rightarrow Y_{0,\bullet}$$

is a Dwyer-Kan equivalence or, equivalently, the map

$$f_{b,\bullet} : X_{b,\bullet} \rightarrow Y_{b,\bullet}$$

is a Dwyer-Kan equivalence for every $b \in B_0$.

3.2. Strictification. Let $F : \mathcal{C}^{op} \rightarrow s\mathcal{S}$ be an simplicially enriched functor. Consider $N\mathcal{C}$ as a bisimplicial space trivially, i.e. $N(\mathcal{C})_{n,m} = N(\mathcal{C})_n$. Form a bisimplicial space $\mathcal{G}(F)$ with space of (m,n) -simplices given by

$$\mathcal{G}(F)_{m,n} = \coprod_{(c_0, \dots, c_m)} \text{map}_{\mathcal{C}}(c_0, c_1) \times \cdots \times \text{map}_{\mathcal{C}}(c_{m-1}, c_m) \times F(c_m)_n$$

We call \mathcal{G} the *Grothendieck construction*. (We point out that this is not the standard usage of the term when applied to functors with values in categories.) There is an adjunction

$$L : ss\mathcal{S}_{/N\mathcal{C}} \rightleftarrows \text{Fun}(\mathcal{C}^{op}, s\mathcal{S}) : \mathcal{G}$$

It is easy to check that

$$L(\Delta[m, n] \xrightarrow{f} N\mathcal{C}) = \Delta[n] \times \text{map}_{\mathcal{C}}(-, c_m)$$

where f corresponds to a string of morphisms $c_0 \rightarrow \cdots \rightarrow c_m$ in \mathcal{C} .

If $F(c)$ is a Segal space for each $c \in \mathcal{C}$, then $\mathcal{G}(F) \rightarrow N\mathcal{C}$ is a Cartesian fibration. The pair (L, \mathcal{G}) lifts to a Quillen pair between the Cartesian model structure on bisimplicial spaces over $N\mathcal{C}$ and the projective model structure on $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_{\infty})$.

Lemma 3.6. *The right adjoint \mathcal{G} preserves weak equivalences.*

Proof. This follows rather directly from the definitions of a weak equivalence in $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_{\infty})$ and the proposition above. \square

When $X = \Delta[m, n]$, the following is essentially the Yoneda lemma.

Lemma 3.7. *For X a cofibrant bisimplicial space and a map $X \rightarrow N\mathcal{C}$, the non-derived unit map $X \rightarrow \mathcal{G}L(X)$ (over $N\mathcal{C}$) is a weak equivalence in the Cartesian fibration model structure.*

Proof. Let X be a representable $f : \Delta[m, n] \rightarrow N\mathcal{C}$ corresponding to a string $c_0 \rightarrow \cdots \rightarrow c_m = c$ of morphisms in \mathcal{C} . Then $L(f) = \Delta[n] \times \text{map}_{\mathcal{C}}(-, c)$ and $\mathcal{G}L(f)$ is the *perpendicular* (or box) product $N(\mathcal{C}/c) \square \Delta[n]$ of $N(\mathcal{C}/c)$ and $\Delta[n]$, i.e. the bisimplicial space with space $N(\mathcal{C}/c)_i \times \Delta[k]_j$ of (i, j) -simplices.

The map $\Delta[0, n] \rightarrow \Delta[m, n]$ (over $N\mathcal{C}$) that selects the object c is a weak equivalence by definition. The map $\Delta[0, n] \rightarrow N(\mathcal{C}/c) \square \Delta[n]$ over $N\mathcal{C}$ is a weak equivalence in the Cartesian fibration model structure as we now explain. Suppose $W \rightarrow N\mathcal{C}$ is a Cartesian fibration. By adjunction, we have

$$\mathbb{R}\text{map}_{N\mathcal{C}}(N(\mathcal{C}/c) \square \Delta[n], W) \simeq \mathbb{R}\text{map}_{N\mathcal{C}}(N(\mathcal{C}/c), W_{\bullet, n})$$

But the right-hand mapping space is weakly equivalent to $\mathbb{R}\mathrm{map}_{NC}(\Delta[0, n], W)$ since $W_{\bullet, n}$ is a right fibration over NC for every n by assumption. This proves the claim for representable objects. For an arbitrary cofibrant X , resolve by representables. \square

Proof of Theorem B. Since \mathcal{G} preserves weak equivalences, the derived unit is weakly equivalent to the non-derived unit for cofibrant objects, which is a weak equivalence by Lemma 3.7. Moreover, the Grothendieck construction \mathcal{G} reflects weak equivalences between fibrant objects.

Both $ss\mathcal{S}_{/NC}$ and $\mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Cat}_\infty)$ are enriched model categories over \mathbf{Cat}_∞ and the pair (L, \mathcal{G}) is a \mathbf{Cat}_∞ -enriched adjunction. Hence, the derived adjunction $(\mathbb{L}L, \mathbb{R}\mathcal{G})$ is an equivalence of $(\infty, 2)$ -categories. \square

Remark 3.8. There is a variant of Theorem B when \mathcal{C} is a bicategory.

4. (CO)LIMITS FOR A GENERAL \mathcal{V}

4.1. Colimits. Let C be a Segal object in \mathcal{V} and $p : X \rightarrow C$ a right fibration.

Proposition 4.1. *The homotopy colimit of p is the homotopy colimit of $X : \Delta^{op} \rightarrow \mathcal{V}$. It is the homotopy left adjoint to the constant diagram functor*

$$s\mathcal{V} \rightarrow s\mathcal{V}_{/C}$$

which sends Y to $Y \times C \rightarrow C$.

Proof. Write $t : X \rightarrow *$ for the terminal map and t^* for the constant diagram functor. Then t^* has a left adjoint $t_!$ given by $t_!(X \rightarrow Z) = X$ and $(t_!, t^*)$ form a Quillen pair for the projective model structure on $s\mathcal{V}$. Because t^* preserves right fibrations, this pair is also Quillen with respect to the right fibration model structures. Note that, for the right fibration model structure on $s\mathcal{V} = s\mathcal{V}_{/*}$, fibrant objects are the homotopically constant simplicial objects (i.e. all maps are weak equivalences).

Write $c : \mathcal{V} \rightarrow s\mathcal{V}$ for the constant diagram functor. It remains to show that $t_!(p) = X$ is weakly equivalent to $c(\mathrm{hocolim} X)$ in the right fibration model structure on $s\mathcal{V}$. For a fibrant (i.e. homotopically constant) simplicial object Z , we have that $c(\mathrm{hocolim} Z) \simeq Z$. So the result follows from the chain of weak equivalences:

$$\begin{aligned} \mathbb{R}\mathrm{map}(X, Z) &\simeq \mathbb{R}\mathrm{map}(X, c(\mathrm{hocolim} Z)) \\ &\simeq \mathbb{R}\mathrm{map}(\mathrm{hocolim} X, \mathrm{hocolim} Z) \\ &\simeq \mathbb{R}\mathrm{map}(c(\mathrm{hocolim} X), c(\mathrm{hocolim} Z)) \end{aligned}$$

where the second equivalence comes from the adjunction between $\mathrm{hocolim}$ and c . \square

4.2. Limits. If \mathcal{V} is a cartesian closed model category, then $s\mathcal{V}$ is a cartesian closed model category with the injective model structure. It follows from the definition that, for every cofibrant object A , the adjunction

$$- \times A : s\mathcal{V} \rightleftarrows s\mathcal{V} : (-)^A$$

forms a Quillen pair. This implies that (and is in fact equivalent to) the following pair is also Quillen:

$$t^* : s\mathcal{V} \rightleftarrows s\mathcal{V}_{/A} : t_*$$

where $t^*(X) = X \times A \rightarrow A$ and $t_*(Y \rightarrow A) = \mathrm{map}_A(A, Y)$, viewed as a constant simplicial object. We then have

Proposition 4.2. *Let B be a (cofibrant) Segal object in \mathcal{V} . The constant diagram functor t^* has a right adjoint*

$$t^* : s\mathcal{V} \rightleftarrows s\mathcal{V}_{/B} : t_*$$

and the pair is Quillen with respect to the right fibration model structure (with underlying injective model structure on $s\mathcal{V}$). Moreover, $\mathbb{R}t_$ evaluated at a right fibration $p : Y \rightarrow B$ is identified with the derived sections object*

$$\mathbb{R}\Gamma(p) := \mathbb{R}\mathrm{map}_B(B, Y)$$

i.e. the pullback of

$$* \rightarrow \mathbb{R}\mathrm{map}(B, B) \leftarrow \mathbb{R}\mathrm{map}(B, Y)$$

computed in \mathcal{V} . We call $\mathbb{R}t_(Y \rightarrow B)$ the (homotopy) limit of Y .*

5. KAN EXTENSIONS

We now deal with Kan extensions of right fibrations with values in spaces. Suppose $i : C \rightarrow D$ is a map between Segal spaces. Base change along i induces a functor

$$i^* : s\mathcal{S}_{/D} \rightarrow s\mathcal{S}_{/C}$$

by setting $i^*(Y \rightarrow D) = Y \times_D C$.

Proposition 5.1 (Left Kan extensions exist). *Base change along i is the right adjoint in a simplicial Quillen pair*

$$i_! : s\mathcal{S}_{/C} \rightleftarrows s\mathcal{S}_{/D} : i^*$$

where both categories are equipped with the right fibration model structure. The left adjoint $i_!$ is called the (homotopy) left Kan extension along i .

Proof. The left adjoint $i_!$ is given by post-composition

$$i_!(Z \xrightarrow{f} C) = Z \xrightarrow{i \circ f} D$$

When $Z = \Delta[n]$, we have

$$\mathrm{map}_D(i_!(f), X) \xrightarrow{\cong} \mathrm{map}_C(f, i^*X)$$

This map is induced, via the Yoneda lemma, by $X_{i(f)} \xrightarrow{\cong} (i^*X)_f$. The general case follows.

By construction, $i_!$ preserves cofibrations since it preserves generating cofibrations. We also know that right fibrations are stable under base change (Proposition 1.11) which implies that i^* preserves fibrant objects (i.e. those right fibrations which are also levelwise fibrations). \square

The next topic is how to characterize of Kan extensions. Let $i : C \rightarrow D$ be a map of Segal spaces. The key construction is a bisimplicial space D/i which in bidegree (n, m) consists of strings of composable morphisms in \mathcal{D} of the form

$$i(c_0) \leftarrow \cdots \leftarrow i(c_m) = d_0 \leftarrow \cdots \leftarrow d_n \leftarrow d$$

(the last m morphisms come from morphisms in C). Functoriality in each simplicial direction is given by composing, or forgetting, morphisms (face maps) or adding identities (degeneracy maps).

More precisely, let us define

$$(D/i)_{n,m} := D_{n+m+1} \times_{D_m} C_m$$

where $D_{n+m+1} \rightarrow D_m$ is the map induced by the inclusion $\gamma := d_{m+1} \dots d_{n+m+1} : [m] \rightarrow [n+m+1]$, i.e. $\gamma(i) = i$.

On morphisms, D/i is constructed as follows. Fix n . The degeneracy maps in the m -direction

$$s_i : (D/i)_{n,m} \rightarrow (D/i)_{n,m+1}$$

(for $0 \leq i \leq m$) are given by (s_{i+n+1}, s_i) . The face maps are defined as (d_{i+n+1}, d_i) (inspection shows that it induces a map of pullbacks). Now fix m . Given a map $j : [n] \rightarrow [n']$ in Δ , extend by the identity to a map $J : [n+m+1] \rightarrow [n'+m+1]$. That is,

$$J(i) = \begin{cases} i & \text{if } i \leq m \\ j(i-m) & \text{if } m \leq i \leq n+m \\ i-n-m & \text{if } i > n+m \end{cases}$$

Then declare

$$(D/i)_{n',m} \rightarrow (D/i)_{n,m}$$

to be the map given by (J^*, id_{C_m}) . We leave to the reader the verification that $(D/i)_{n,m}$ as defined is indeed a bisimplicial space.

There is an alternative definition of D/i which highlights functoriality. Namely, for $n \geq 0$, let $(D/i)_n$ be the simplicial space defined as the homotopy limit of the diagram

$$D_n \times \Delta[0] \rightarrow D^{\Delta[n]} \xleftarrow{d_0} D^{\Delta[n+1]} \xrightarrow{d_1 \dots d_{n-1}} D \leftarrow C$$

(recall that $d^1 \dots d^{n-1} : [0] \rightarrow [n]$ is the ultimate target map). This is clearly functorial in $[n]$. Under the assumption that C and D are Segal spaces, it is not hard to see that the two definitions of D/i coincide.

Note that, for each $m \geq 0$, there is a map of simplicial spaces

$$(5.1) \quad (D/i)_{\bullet,m} \rightarrow D$$

induced by $\beta^* : D_{n+m+1} \rightarrow D_n$, where $\beta : [n] \rightarrow [n+m+1]$ sends i to $m+1+i$. With the second description of D/i , this map is simply the projection onto the $D_n \times \Delta[0]$ factor.

Lemma 5.2. *For each $m \geq 0$, (5.1) is a right fibration.*

Proof. Since D is a Segal space, we have that $D_n \times_{D_0} \times D_{m+1} \simeq D_{n+m+1}$, and so it follows that $D_n \times_{D_0} (D/i)_{0,m} \simeq (D/i)_{n,m}$. \square

Proposition 5.3. *Let X be a right fibration over C . The simplicial space given by*

$$[n] \mapsto \operatorname{hocolim}_{[m] \in \Delta} (D/i)_{n,m} \times_{C_m} X_m$$

is a right fibration over D and computes $\mathbb{L}i_!(X)$, the homotopy left Kan extension of X along i .

Proof. For the duration of this proof, we will sometimes write \mathcal{L} for the simplicial space over D in the statement of the proposition.

Because in \mathcal{S} homotopy colimits are stable under homotopy base change, we have

$$(\operatorname{hocolim}_{\Delta} (D/i)_n \times_C X) \times_{D_0}^h D_1 \simeq \operatorname{hocolim}_{\Delta} ((D/i)_n \times_{D_0}^h D_1 \times_C X)$$

(here we have suppressed m from the notation). By Lemma 5.2, the target is weakly equivalent to

$$\operatorname{hocolim}_{\Delta} ((D/i)_{n+1} \times_C^h X)$$

and so, by proposition 1.7, we conclude that $\mathcal{L} \rightarrow D$ is a right fibration.

The second claim is that there is a natural zigzag of weak equivalences relating $\mathbb{L}i_!(X)$ and \mathcal{L} in the right fibration model structure.

We may reduce statement to representables, i.e. X of the form $C/K \rightarrow C$ for $K \subset C_0$. We take $K = C_0$ (the general case is analogous). In that case, $\mathbb{L}i_!(X)$ is identified with $i_0 : \Delta[0] \times C_0 \rightarrow D$ – first, using the Yoneda lemma, we replace X with $\Delta[0] \times C_0 \rightarrow C$, and then we apply (non-derived) $i_!$.

We are left to identify $i_0 : \Delta[0] \times C_0 \rightarrow D$ and \mathcal{L} . Applying the Yoneda lemma again, we identify $i_0 : \Delta[0] \times C_0 \rightarrow D$ with

$$D/C_0 \rightarrow D$$

Note that $(D/C_0)_n = (D/i)_{n,0}$. In the remainder of the proof, we will construct a weak equivalence³

$$\operatorname{hocolim}_{[m] \in \Delta} (D/i)_{n,m} \times_{C_m} C_{m+1} \simeq (D/i)_{n,0}$$

The claimed weak equivalence will be in fact given as a zigzag of weak equivalences. As a first step, we can replace the hocolim in the display above with

$$(5.2) \quad \operatorname{hocolim}_{[m] \in \Delta} (D/i)_{n,m+1}$$

Indeed, there is a map of simplicial objects

$$(5.3) \quad (D/i)_{n,m+1} \rightarrow (D/i)_{n,m} \times_{C_m} C_{m+1}$$

induced by the map between homotopy pullbacks of the rows in the diagram

$$\begin{array}{ccccc} D_{n+m+2} & \xrightarrow{\gamma^*} & D_{m+1} & \xleftarrow{p} & C_{m+1} \\ d_0 \downarrow & & \downarrow d_0 & & \downarrow id \\ D_{n+m+1} & \xrightarrow{\gamma^*} & D_m & \xleftarrow{d_0 p} & C_{m+1} \end{array}$$

(recall γ is the map $\gamma(i) = i$, wherever defined). But the left-hand square in the diagram is homotopy cartesian since D is a Segal space, and it follows that the induced map on homotopy pullbacks is a weak equivalence. Thus, (5.3) is a levelwise weak equivalence of bisimplicial spaces.

Now, the face operator $d_1 : D_{n+2} \rightarrow D_{n+1}$ (which corresponds to the composition of the last two morphisms) yields a map

$$(D/i)_{n,1} \rightarrow (D/i)_{n,0}$$

which is an augmentation for $[m] \mapsto (D/i)_{n,m+1}$, and thus gives rise to a map from (5.2) to $(D/i)_{n,0}$. It remains to show that this map is a levelwise weak equivalence of simplicial spaces (in the variable n). Since both simplicial spaces are right fibrations over D , it is enough to show that it is a weak equivalence on 0-simplices, i.e. for $n = 0$. This can be done by exhibiting a contracting homotopy using the extra degeneracy maps provided by $s_0 : D_k \rightarrow D_{k+1}$. \square

When $D = *$, we recover

³This can be regarded as an analogue of the Yoneda lemma in the following formulation:

$$\operatorname{map}_{\mathcal{D}}(-, i(-)) \otimes_{\mathcal{C}} \operatorname{map}_{\mathcal{C}}(-, c) \cong \operatorname{map}_{\mathcal{D}}(-, i(c))$$

Corollary 5.4. *Let $p : X \rightarrow C$ be a right fibration. The homotopy colimit of p (as a right fibration) is the homotopy colimit of $X : \Delta^{op} \rightarrow \mathcal{S}$.*

Remark 5.5. When i is (the nerve of) a functor of \mathcal{S} -categories and F is a presheaf on C , $i_! \mathcal{G}(F)$ agrees with usual left Kan extension of F along i . In particular, when D is a point, $i_! \mathcal{G}(F)$ computes the homotopy colimit of F . In that case, the simplicial space X is the Grothendieck construction of the functor F and it is often Reedy cofibrant (e.g. it is always so if \mathcal{C} is discrete) in which case the homotopy colimit agrees with the geometric realization and we recover the usual Bousfield-Kan formulae.

5.1. Invariance under ∞ -categorical equivalences.

Proposition 5.6. *If $i : C \rightarrow D$ is a Dwyer-Kan equivalence between Segal spaces, then*

$$i_! : s\mathcal{S}/_C \xrightarrow{\sim} s\mathcal{S}/_D : i^*$$

is a simplicial Quillen equivalence, where both sides have the right fibration model structure.

Proof. We show that the derived unit and counit are weak equivalences. Let $X \in s\mathcal{S}/_C$. The derived unit is the morphism (over C)

$$(5.4) \quad X \rightarrow \mathbb{R}i^* \mathbb{L}i_! X$$

We begin with the case $X = \{c\} : \Delta[0] \xrightarrow{c} C$. By the Yoneda lemma (Lemma 1.31), the vertical maps in the diagram below are weak equivalences in the right fibration model structure

$$\begin{array}{ccc} \{c\} & \longrightarrow & \mathbb{R}i^* \mathbb{L}i_!(\{c\}) \\ \simeq \downarrow & & \downarrow \simeq \\ C/c & \longrightarrow & \mathbb{R}i^* \mathbb{L}i_!(C/c) \end{array}$$

But $i_!(\{c\}) = D/i(c)$ and $\{c\}$ is cofibrant, and so $\mathbb{R}i^* \mathbb{L}i_!(\{c\}) \simeq \mathbb{R}i^*(D/i(c))$. Moreover, $D/i(c) \rightarrow D$ is a right fibration, so $\mathbb{R}i^*(D/i(c)) \simeq i^*(D/i(c))$. This shows that the top right-hand space in the square is weakly equivalent to $i^*(D/i(c))$.

Now, by hypothesis i is fully faithful. That is to say, the map $C/c \rightarrow i^*(D/i(c))$ is a weak equivalence. This shows that the horizontal maps in the diagram above are weak equivalences.

When X is of the form $f : \Delta[n] \rightarrow C$ corresponding to a string of maps $c_1 \rightarrow \cdots \rightarrow c_n$, the claim follows from the previous case by Proposition 1.7. The general case can be proved by reduction to the previous two cases, in the same spirit as the proofs of the previous propositions. Given a simplicial space X over C , resolve it by representables Y_\bullet . Since $i_!$ and i^* commute with colimits (the latter since it is a left adjoint, the former since colimits in spaces are stable under base change), the map (5.4) is a weak equivalence if

$$Y_k \rightarrow \mathbb{R}i^* \mathbb{L}i_! Y_k$$

is a weak equivalence for each $k \geq 0$. This concludes the proof that the derived unit is a weak equivalence.

It remains to show that the derived counit

$$(5.5) \quad \mathbb{L}i_! \mathbb{R}i^* Z \rightarrow Z$$

is a weak equivalence, for each $Z \in s\mathcal{S}_{/D}$.

Suppose first that Z is of the form D/d for some $d \in D_0$. Then Z is fibrant, so the source of the counit map is weakly equivalent to $\mathbb{L}i_!i^*D/d$. Since, by hypothesis, i is essentially surjective, there exists some $c \in C$ such that $i(c) \simeq d$ and consequently $i^*D/d \simeq i^*D/i(c)$. Since i is also fully faithful, we conclude that i^*D/d is weakly equivalent to C/c in $s\mathcal{S}_{/C}$. To finish the argument, we need to use the fact that $\{c\} \simeq C/c$ by Lemma 1.31, and that $\{c\} := \Delta[0] \xrightarrow{c} C$ is cofibrant. Hence,

$$\mathbb{L}i_!i^*D/d \rightarrow i_!(\{c\})$$

is a weak equivalence in $s\mathcal{S}_{/D}$. The target of this map is by definition $\{i(c)\}$ (notation for $i(c) : \Delta[0] \rightarrow D$) which is, again by Proposition 1.30, weakly equivalent to D/d .

Given an arbitrary $Z \in s\mathcal{S}_{/D}$, resolve it by cofibrant objects. These will be of the form $D/d \times K$ for some space K . There are at least two ways to make this reduction. One is to resolve Z by representable simplicial spaces and apply Lemma 1.31. The other is to strictify Y to a strict presheaf $L(Y)$, resolve $L(Y)$ by representable presheaves, and apply the Grothendieck construction \mathcal{G} to the simplicial resolution to obtain a resolution of $Y \simeq \mathcal{G}L(Z)$ (Lemma 1.36). In each case, one obtains a simplicial object where each simplicial level is a coproduct of simplicial spaces of the aforementioned form. \square

Corollary 5.7. *Suppose B is a Segal space (not necessarily complete) and let $i : B \rightarrow B^\sharp$ be its Rezk completion, i.e. a Dwyer-Kan equivalence to a complete Segal space B^\sharp . Then i^* is a Quillen equivalence between the right fibration model structures on B and B^\sharp .*

Remark 5.8. Since, by [Ber07], any Segal space B may be strictified (i.e. is Dwyer-Kan equivalent) to some simplicially enriched category \mathcal{C} , Theorem A and the invariance statement above together imply that any right fibration over B may be strictified to a space-valued presheaf on \mathcal{C} .

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